

ENERGY-MINIMAL DIFFEOMORPHISMS BETWEEN DOUBLY CONNECTED RIEMANN SURFACES

DAVID KALAJ

ABSTRACT. Let M and N be doubly connected Riemann surfaces with boundaries and with nonvanishing conformal metrics σ and ρ respectively, and assume that ρ is a smooth metric with bounded Gauss curvature \mathcal{K} and finite area. The paper establishes the existence of homeomorphisms between M and N that minimize the Dirichlet energy. *In the class of all homeomorphisms $f: M \xrightarrow{\text{onto}} N$ between doubly connected Riemann surfaces such that $\text{Mod } M \leq \text{Mod } N$ there exists, unique up to conformal automorphisms of M , an energy-minimal diffeomorphism which is a harmonic diffeomorphism.* The results improve and extend some recent results of Iwaniec, Koh, Kovalev and Onninen (Inven. Math. (2011)), where the authors considered bounded doubly connected domains in the complex plane w.r. to Euclidean metric.

1. INTRODUCTION

The primary goal of this paper is to study the minimum of energy integral of mappings defined between annuli in Riemann surfaces. We will study the existence of a diffeomorphism $f: \Omega \xrightarrow{\text{onto}} \Omega^*$ of smallest (finite) ρ -Dirichlet energy (see (1.3)) where ρ is an arbitrary smooth metric with bounded Gauss curvature and finite area defined on Ω^* . We extend in this way some results obtained in a recent essential paper of Iwaniec, Koh, Kovalev and Onninen ([15]) where the authors considered the same problem but for planar case. We will follow closely the ideas and notation of the paper [15] but we need to make a new approach, due to the different geometry. We begin by the definition of harmonic mappings.

1.1. Harmonic mappings between Riemann surfaces. Let M and N be Riemann surfaces with metrics σ and ρ , respectively. If a mapping $f: M \rightarrow N$, is C^2 , then f is said to be harmonic (to avoid the confusion we will sometimes say ρ -harmonic) if

$$(1.1) \quad f_{z\bar{z}} + (\log \rho^2)_w \circ f f_z f_{\bar{z}} = 0,$$

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where z and w are the local parameters on M and N respectively. Also f satisfies (1.1) if and only if its Hopf differential

$$(1.2) \quad \text{Hopf}(f) = \rho^2 \circ f f_z \overline{f_{\bar{z}}}$$

is a holomorphic quadratic differential on M . Let

$$|\partial f|^2 := \frac{\rho^2(f(z))}{\sigma^2(z)} \left| \frac{\partial f}{\partial z} \right|^2 \quad \text{and} \quad |\bar{\partial} f|^2 := \frac{\rho^2(f(z))}{\sigma^2(z)} \left| \frac{\partial f}{\partial \bar{z}} \right|^2$$

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are standard complex partial derivatives. The ρ -Jacobian is defined by

$$J(f) := |\partial f|^2 - |\bar{\partial} f|^2.$$

If u is sense preserving, then the ρ -Jacobian is positive. The Hilbert-Schmidt norm of differential df is the square root of the energy $e(f)$ and is defined by

$$(1.3) \quad |df| = \sqrt{2|\partial f|^2 + 2|\bar{\partial} f|^2}.$$

For $g : M \mapsto N$ the ρ -Dirichlet energy is defined by

$$(1.4) \quad \mathcal{E}^\rho[g] = \int_M |dg|^2 dV_\sigma,$$

where ∂g , and $\bar{\partial} g$ are the partial derivatives taken with respect to the metrics ϱ and σ , and dV_σ , is the volume element on (M, σ) , which in local coordinates takes the form $\sigma^2(w) du \wedge dv$, $w = u + iv$. Assume that energy integral of f is bounded. Then a stationary point f of the corresponding functional where the homotopy class of f is the range of this functional is a harmonic mapping. The converse is not true. More precisely there exists a harmonic mapping which is not stationary. The literature is not always clear on this point because for many authors, a harmonic mapping is a stationary point of the energy integral. For the last definition and some important properties of harmonic maps see [34]. It follows from the definition that, if a is conformal and f is harmonic, then $f \circ a$ is harmonic. Moreover if b is conformal, $b \circ f$ is also harmonic but with respect to (possibly) an another metric ρ_1 (see Lemma 2.3 below).

Moreover if N and M are double connected Riemann surfaces with non vanishing metrics σ and ρ , then by [16, Theorem 3.1], there exist conformal mappings $X : \Omega \xrightarrow{\text{onto}} M$ and $X^* : \Omega^* \xrightarrow{\text{onto}} N$ between double connected plane domains Ω and Ω^* and Riemann surfaces M and N respectively.

Notice that the harmonicity neither Dirichlet energy do not depend on metric σ on domain so we will assume from now on $\sigma(z) \equiv 1$. This is why throughout this paper $M = (\Omega, \mathbf{1})$ and $N = (\Omega^*, \rho)$ will be doubly connected domains in the complex plane \mathbb{C} (possibly unbounded), where $\mathbf{1}$ is the Euclidean metric. Moreover ρ is a nonvanishing smooth metric defined in Ω^* with bounded Gauss curvature \mathcal{K} where

$$(1.5) \quad \mathcal{K}(z) = -\frac{\Delta \log \rho(z)}{\rho^2(z)},$$

(we put $\kappa := \sup_{z \in \Omega^*} |\mathcal{K}(z)| < \infty$) and with finite area defined by

$$\mathcal{A}(\rho) = \int_{\Omega^*} \rho^2(w) du dv, \quad w = u + iv.$$

We call such a metric *allowable* one (cf. [1, P. 11]). If ρ is a given metric in Ω^* , we conventionally extend it to be equal to 0 in $\partial\Omega^*$. As we already pointed out, we will study the minimum of Dirichlet integral of mappings between certain sets. We refer to introduction of [15] and references therein for good setting of this problem and some connection with the theory of nonlinear elasticity. Notice first that a change of variables $w = f(z)$ in (1.4) yields

$$(1.6) \quad \mathcal{E}^\rho[f] = 2 \int_{\Omega} \rho^2(f(z)) J_f(z) dz + 4 \int_{\Omega} \rho^2(f(z)) |f_{\bar{z}}|^2 dz \geq 2\mathcal{A}(\rho)$$

where J_f is the Jacobian determinant and $\mathcal{A}(\rho)$ is the area of Ω^* and $dz := dx \wedge dy$ is the area element w.r. to Lebesgue measure on the complex plane. A conformal mapping of $f : \Omega \xrightarrow{\text{onto}} \Omega^*$; that is, a homeomorphic solution of the Cauchy-Riemann system $f_{\bar{z}} = 0$, would be an obvious choice for the minimizer of (1.6). For arbitrary multiply connected domains there is no such mapping.

In the case of Euclidean metric $\rho \equiv 1$, the existence of a harmonic diffeomorphism between certain sets does not imply the existence of an energy-minimal one, see [15, Example 9.1]. Example 9.1 in [15] has been constructed with help of affine self-mappings of the complex plane. For a general metric ρ , affine transformations are not harmonic, thus we cannot produce a similar example.

First of all, energy-minimal diffeomorphisms for bounded simply connected domains exist by virtue of the Riemann mapping theorem. We continue to study the doubly connected case which has been already studied before for some special cases. The case of circular annuli w.r. to Euclidean metric and the metric $\rho(w) = 1/|w|$ is established in [2]. This result has been extended to all radial metrics in [22]. Recently in [15], by using very interesting approach the authors established the case of bounded doubly connected domains in the complex plane w.r. to Euclidean metric, provided that the domain has smaller modulus than the target [15, Theorem 1.1].

1.2. Statement of results. One of the main results of this paper is the following generalization of the main result in [15].

Theorem 1.1. *Suppose that Ω and Ω^* are doubly connected domains in \mathbb{C} such that $\text{Mod } \Omega \leq \text{Mod } \Omega^*$ and let ρ be an allowable metric in Ω^* . Then there exists an ρ -energy-minimal diffeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega^*$, which is unique up to a conformal change of variables in Ω .*

In particular case of the Euclidean metric, our results improve the main result in [15] because in contrast to [15] in this paper we relax from the assumption that Ω is a bounded domain. For the Euclidean metric, the

target Ω^* cannot be arbitrary, because the energy of every diffeomorphism $f: \Omega \mapsto \Omega^*$ is larger than the area $|\Omega^*|$ of Ω^* . In other words the Euclidean metric is allowable for every domain Ω^* with finite area (not necessarily bounded).

The notation $\text{Mod } \Omega$ stands for the *conformal modulus* of Ω . Any doubly connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C} \setminus \{a\}$ is conformally equivalent to some *circular annulus*

$$A(\tau) := A(e^{-\tau}, 1) = \{z: e^{-\tau} < |z| < 1\}$$

with $0 < \tau \leq \infty$. The number τ defines $\text{Mod } \Omega := \tau$. By virtue of [16, Theorem 3.1] the modulus can be defined uniquely for double connected Riemann surfaces with boundaries as well. The conformal modulus is infinite precisely when one of components of $\mathbb{C} \setminus \Omega$ degenerates to a point. We call such domain a *punctured domain*. Theorem 1.1 has the following corollary.

Corollary 1.2. For any doubly connected domain Ω and any punctured domain Ω^* with allowable metric ρ there exists an energy-minimal diffeomorphism $f: \Omega \xrightarrow{\text{ontq}} \Omega^*$, which is unique up to a conformal change of variables in Ω .

The main subject of classical Teichmüller theory is the existence of quasiconformal mappings $g: \Omega^* \xrightarrow{\text{ontq}} \Omega$ with smallest L^∞ -norm of the distortion function

$$K_g(w) = \frac{|Dg(w)|^2}{2J_g(w)}, \quad \text{a.e. } w \in \Omega^*$$

see [1]. Let $d\mu(w) = \rho^2(w)dudv$ be the measure at Ω^* and let $L^1 = L^1(\Omega^*, \mu)$ be the Lebesgue space of integrable functions defined in Ω^* (the norm is defined by

$$\|\Phi\|_{L^1} := \int_{\Omega^*} \rho^2(w)|\Phi(w)|dudv,$$

where $w = u + iv$). Then the L^1 -norm of K_g and Dirichlet energy of the inverse mapping are related via the transformation formula

$$(1.7) \quad \|K_g\|_{L^1} = \mathcal{E}^\rho[f], \quad \text{where } f = g^{-1}: \Omega \xrightarrow{\text{ontq}} \Omega^*.$$

For rigorous statements let us recall that a homeomorphism $g: \Omega^* \xrightarrow{\text{ontq}} \Omega$ of Sobolev class $W_{loc}^{1,1}(\Omega^*)$ has integrable distortion if

$$(1.8) \quad |Dg(w)|^2 \leq 2K(w)J_g(w) \quad \text{a.e. in } \Omega^*$$

for some $K \in L^1(\Omega^*, \mu)$. The smallest such $K_g: \Omega^* \rightarrow [1, \infty)$, denoted by K_g , is referred to as the distortion function of g .

It turns out that the inverse of any mapping with integrable distortion has finite Dirichlet energy and the identity (1.7) holds. As a consequence of Theorem 1.1 we obtain the following result.

Theorem 1.3. Suppose that Ω and Ω^* are doubly connected domains in \mathbb{C} such that $\text{Mod } \Omega \leq \text{Mod } \Omega^*$ and ρ is an allowable metric in Ω^* . Then, among

all homeomorphisms $g: \Omega^* \xrightarrow{\text{onto}} \Omega$ there exists, unique up to a conformal automorphism of Ω , a mapping of smallest $L^1 = L^1(\Omega^*, \mu)$ norm of the distortion. Here $d\mu(w) = \rho^2(w)du \wedge dv$.

We continue this introduction with a brief outline of the proof of Theorem 1.1 i.e. of its refined variant Theorem 1.4. The natural set for our minimization problem is the subset $W_l^{1,2}$ of local Sobolev space $W_{loc}^{1,2}(\Omega, \bar{\Omega}^*)$ of maps h whose first derivative, i.e. the function $|dh|$ (see (1.3)) lies in $L^2(\Omega)$. In this paper functions in the Sobolev spaces are complex-valued. Let us reserve the notation $H^{1,2}(\Omega, \Omega^*)$ for the set of all sense-preserving $W_l^{1,2}$ -homeomorphisms $h: \Omega \xrightarrow{\text{onto}} \Omega^*$. When this set is nonempty, we define

$$(1.9) \quad E_H^\rho(\Omega, \Omega^*) = \inf\{\mathcal{E}^\rho[h]: h \in H^{1,2}(\Omega, \Omega^*)\}.$$

It is important to note that $W_l^{1,2}$ and the class $H^{1,2}(\Omega, \Omega^*)$ depend on the metric ρ .

A homeomorphism $h \in H^{1,2}(\Omega, \Omega^*)$ is *energy-minimal* if it attains the infimum in (1.9). Let us notice that the set $H^{1,2}(\Omega, \Omega^*) \subset W_l^{1,2}$ is not bounded neither bounded subsets of $H^{1,2}(\Omega, \Omega^*)$ are compact.

In order to minimize the energy integral, similarly as in [15], we introduce the class of so-called *deformations*. These are sense-preserving surjective mappings of the Sobolev class $W_l^{1,2}$ that can be approximated by homeomorphisms (diffeomorphisms) in a certain way which make itself a compact family of mappings. The precise definition of the class of deformations $\mathfrak{D}^\rho(\Omega, \Omega^*)$ is given in §2. Notice that $H^{1,2}(\Omega, \Omega^*) \subset \mathfrak{D}^\rho(\Omega, \Omega^*)$. A deformation is not necessarily injective neither bounded. Define

$$(1.10) \quad E^\rho(\Omega, \Omega^*) = \inf\{\mathcal{E}^\rho[h]: h \in \mathfrak{D}^\rho(\Omega, \Omega^*)\}$$

where $\mathcal{E}^\rho[h]$ is as in (1.4). A deformation that attains the infimum in (1.10) is called *energy-minimal*. It is obvious that $E_H^\rho(\Omega, \Omega^*) \geq E^\rho(\Omega, \Omega^*)$, but whether the equality holds is not clear. The following theorem will imply Theorem 1.1.

Theorem 1.4. *Suppose that Ω and Ω^* are doubly connected domains in \mathbb{C} such that $\text{Mod } \Omega \leq \text{Mod } \Omega^*$ and let ρ be an allowable metric in Ω^* . Then there exists a diffeomorphism $h \in H^{1,2}(\Omega, \Omega^*)$ that minimizes the energy among all deformations; that is,*

$$\mathcal{E}^\rho[h] = E^\rho(\Omega, \Omega^*)$$

and hence,

$$E_H^\rho(\Omega, \Omega^*) = E^\rho(\Omega, \Omega^*).$$

Behind our approach is the calculus of variation. We rely on certain inner variations, which yield, according to a trivial modification of a result of Jost [17], that the Hopf differential (§5) of an energy-minimal deformation w.r. to these variation is holomorphic in Ω and real on its boundary. Our inner variations will prove efficient. Indeed, we will show later that the energy-minimal deformation exists and is unique, up to a conformal automorphisms

of Ω , under the condition $\text{Mod}(\Omega) \leq \text{Mod}(\Omega^*)$. By using the result of Jost and improved Reich-Walczak-type inequalities (§4), as in [15], we obtain additional information about the Hopf differential where the conformal moduli of Ω and Ω^* play their roles.

As in [15], we consider a one-parameter family of variational problems in which Ω changes continuously while Ω^* (and ρ) remain fixed. This was crucial idea in [15] and works in this setting as well. We establish strict monotonicity of the minimal energy as a function of the conformal modulus of Ω (§6) (without additional assumption on the boundary of target) improving in this way the corresponding result in [15]. The proof of Theorem 1.4 (and consequently of Theorem 1.1), is completed in §7. In §8 we establish the strict convexity of the minimal energy provided that the modulus of domain is smaller than the modulus of target. In the final section §9 we consider the special situation where Ω^* is a circular annulus with radial metric. We prove the strict convexity of minimal energy under the so-called Nitsche condition. In addition we prove that under the Nitsche condition the minimal energy is attained for quasiconformal harmonic mappings which are shown to be natural generalization of conformal mappings.

Remark 1.5. The existence of a harmonic diffeomorphisms with prescribed boundary data (which solve the Dirichlet problem) between certain simply connected domains in Riemann surfaces was primary purpose of a large number of papers and books. We refer to the monographs of Jost [16] and Hamilton [5]. On the other hand, as is observed by J. C. C. Nitsche [31], if the domains are doubly connected (in particular circular annuli), then the existence of a harmonic diffeomorphism between them is not assured even though we do not take care of boundary data. In this paper, as a by-product of main theorem (Theorem 1.4) it is established the existence of ρ -harmonic diffeomorphisms between doubly connected domains Ω and Ω^* in \mathbb{C} provided that $\text{Mod } \Omega \leq \text{Mod } \Omega^*$ and ρ is an allowable metric. Notice that, the same approach can be deduced for arbitrary doubly connected domains Ω and Ω^* in Riemann surfaces (M, σ) and (N, ρ) . This is a variation of the corresponding result [13] where the authors settled the same problem but for the Euclidean metric ($\rho(w) \equiv 1$). Indeed they obtained a better result in this special case by using the fact that the Euclidean harmonicity is invariant under affine transformations of the target, which is not the case for arbitrary ρ -harmonic mappings.

In opposite direction, for every $\epsilon > 0$ there exists a pair of smooth bounded doubly connected domains Ω, Ω^* with $0 < \text{Mod } \Omega - \Psi_\rho(\text{Mod } \Omega^*) < \epsilon$, where

$$\Psi_p = \Psi_\rho(\omega) := \int_R^1 \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - R^2\rho^2(R)}}$$

for which there is no energy-minimal homeomorphism in $H^{1,2}(\Omega, \Omega^*)$ (See Section 9 and the reference [22]).

2. DEFORMATIONS

A homeomorphism of a planar domain is either sense-preserving or sense-reversing. For homeomorphisms of the Sobolev class $W_{loc}^{1,1}(\Omega, \Omega^*)$ this implies that the Jacobian determinant does not change sign: it is either non-negative or nonpositive at almost every point [3, Theorem 3.3.4], see also [7]. The homeomorphisms considered in this paper are sense-preserving.

Let Ω and Ω^* be domains in \mathbb{C} . Let $a \in \Omega^*$ and $b \in \partial\Omega^*$. We define

$$\text{dist}_\rho(a, b) := \inf_\gamma \int_\gamma \rho(w) |dw|,$$

where γ ranges over all rectifiable Jordan arcs connecting a and b within Ω^* if the set of such Jordan arcs is not empty (otherwise we conventionally put $\text{dist}_\rho(a, b) = \infty$). To every mapping $f: \Omega \rightarrow \overline{\Omega^*}$ we associate a boundary distance function

$$\delta_f^\rho(z) = \text{dist}_\rho(f(z), \partial\Omega^*) = \inf_{b \in \partial\Omega^*} \text{dist}_\rho(f(z), b)$$

which is set to 0 on the boundary of Ω .

We now adapt for our purpose the concepts of $c\delta$ -uniform convergence and of deformation defined for Euclidean metric and bounded domains in [15].

Definition 2.1. A sequence of continuous mappings $h_j: \Omega \rightarrow \Omega^*$ is said to converge *$c\delta$ -uniformly* to $h: \Omega \rightarrow \Omega^*$ if

- (1) $h_j \rightarrow h$ uniformly on compact subsets of Ω and
- (2) $\delta_{h_j}^\rho \rightarrow \delta_h^\rho$ uniformly on $\overline{\Omega}$.

We designate it as $h_j \xrightarrow{c\delta} h$. Concerning the item (1) we need to notice that the Euclidean metric and the nonvanishing smooth metric ρ are equivalent on compacts of Ω^* .

Definition 2.2. A mapping $h: \Omega \rightarrow \overline{\Omega^*}$ is called a ρ *deformation* (or just deformation) if

- (1) $h \in W_{loc}^{1,2}$ and $|dh| \in L^2$ (which we write shortly $h \in W_l^{1,2}$);
- (2) The Jacobian $J_h := \det Dh$ is nonnegative a.e. in Ω ;
- (3) $\int_\Omega \rho^2(h(z)) J_h \leq \mathcal{A}(\rho)$;
- (4) there exist sense-preserving diffeomorphisms $h_j: \Omega \xrightarrow{\text{onto}} \Omega^*$, called an *approximating sequence*, such that $h_j \xrightarrow{c\delta} h$ on Ω .

The set of ρ deformations $h: \Omega \rightarrow \overline{\Omega^*}$ is denoted by $\mathfrak{D}^\rho(\Omega, \Omega^*)$.

Notice first that the condition (1) of the previous definition in the case of bounded domains Ω and Ω^* can be replaced by $h \in W^{1,2}$ see Lemma 2.3.

Notice also that $W_l^{1,2}$ depends on ρ because the Hilbert-Schmidt norm depends on ρ . In order to compare Definition 2.2 with the corresponding definition [15, Definition 2.2] prove that, they coincide if ρ is the Euclidean metric, and Ω and Ω^* are doubly connected bounded domains. We need to show that, for a given approximating sequence of homeomorphisms there

exists an approximating sequence of diffeomorphisms of h_0 . For doubly connected domains Ω and Ω^* , there exist conformal mappings $a : A \xrightarrow{\text{ontq}} \Omega$ and $b : \Omega^* \xrightarrow{\text{ontq}} A^*$ such that, A and A^* are circular annuli. Now for a given homeomorphism $h : \Omega \xrightarrow{\text{ontq}} \Omega^*$, the mapping $\tilde{h} = b \circ h \circ a$ is a homeomorphism between A and A^* . We can map A and A^* by means of diffeomorphisms χ and χ^* to $\mathbb{S}^2 \setminus \{N, S\}$, where N and S are north and south pole of the unit sphere $\mathbb{S}^2 \subset \mathbf{R}^3$. This induces a unique homeomorphism $\tilde{\tilde{h}} : \mathbb{S}^2 \xrightarrow{\text{ontq}} \mathbb{S}^2$. By a T. Rado's theorem, the homeomorphism $\tilde{\tilde{h}}$ can be approximated uniformly by a sequence of diffeomorphisms $\tilde{\tilde{h}}_j$ leaving the points N and S fixed. This induces a sequence of diffeomorphisms $h_j : \Omega \xrightarrow{\text{ontq}} \Omega^*$ converging $c\delta$ -uniformly to h . The diagonal selection will produce the desired approximating sequence of h_0 .

Further we have that $H^{1,2}(\Omega, \Omega^*) \subset \mathcal{D}^\rho(\Omega, \Omega^*)$. Outside of some degenerate cases, the set of deformations is nonempty by Lemma 2.16 and is closed under weak limits in $W^{1,2}(\Omega)$ by Lemma 2.1.

2.1. Some properties of deformations. From now on we assume that ρ is an allowable metric in Ω^* . We begin with the following lemma which will simplify our approach throughout the paper.

Lemma 2.3. *Assume that Ω and Ω^* are doubly connected domains, and assume that $a : \Omega \xrightarrow{\text{ontq}} A(\tau)$ and $b : A(\omega) \xrightarrow{\text{ontq}} \Omega^*$ are univalent conformal mappings and define $\rho_1(w) = \rho(b(w))|b'(w)|$, $w \in A(\omega)$. Then*

- (a) $\mathcal{E}^\rho[b \circ f \circ a] = \mathcal{E}^{\rho_1}[f]$ provided that one of the two sides exist.
- (b) $b \circ f \circ a \in D^\rho(\Omega, \Omega^*)$ if and only if $f \in D^{\rho_1}(A(\tau), A(\omega))$.
- (c) For Gauss curvature we have $\mathcal{K}_\rho(b(w)) = \mathcal{K}_{\rho_1}(w)$.
- (d) ρ is an allowable metric if and only if ρ_1 is an allowable metric.
- (e) $W_l^{1,2}(A(\tau), A(\omega)) = W^{1,2}(A(\tau), A(\omega))$.
- (f) $b \circ f \circ a$ is ρ -harmonic if and only if f is ρ_1 -harmonic.

Proof. Let us show (a). By using the change of variables $w = a(z)$ we obtain

$$\begin{aligned}
 \mathcal{E}^{\rho_1}[f] &= \int_{A(\tau)} |df|^2 = \int_{A(\tau)} \rho_1^2(f(w)) |Df|^2 \\
 &= \int_{A(\tau)} \rho^2(b(f(w))) |b'(f(w))|^2 |Df(w)|^2 \\
 &= \int_{\Omega} \rho^2(f(a(z))) |b'(f(a(z)))|^2 |Df(a(z))|^2 a'(z) \\
 &= \int_{\Omega} |d(b \circ f \circ a)|^2 = \mathcal{E}^\rho[b \circ f \circ a].
 \end{aligned}$$

Prove now (b). First of all from (a) we have $|d(b \circ f \circ a)| \in L^2$ if and only if $|d(f)| \in L^2$. Further $J_{b \circ f \circ a} = |b'|^2 J_f |a'|^2$ and therefore $J_{b \circ f \circ a} \geq 0$ if and only if $J_f \geq 0$. The equality $\mathcal{A}(\rho) = \mathcal{A}(\rho_1)$ follows by using the change of

variables $w = b(z)$ in the integral $\int_{\Omega^*} \rho^2(w) dw$. Furthermore

$$\int_{\Omega} \rho^2(b \circ f \circ a(z)) J_{b \circ f \circ a} = \int_{A(\tau)} \rho_1^2(f(z)) J_f.$$

If h_j is the corresponding approximating sequence of diffeomorphisms for the deformation f then the sequence $b \circ h_j \circ a$ corresponds to the deformation $b \circ f \circ a$. The item (c) follows from formula for Gauss curvature (1.5). The equality $W_l^{1,2}(A(\tau), A(\omega)) = W^{1,2}(A(\tau), A(\omega))$ follows from the fact that every mapping f between $A(\tau)$ and $A(\omega)$ is bounded, and consequently the inequality

$$\|f\|_{W^{1,2}}^2 := \int_{A(\tau)} (|f|^2 + |df|^2) < \infty$$

is automatically satisfied if $\int_{A(\tau)} |df|^2 < \infty$. The item (f) follows from the fact that Hopf differential of f is holomorphic if and only if Hopf differential of $b \circ f \circ a$ is holomorphic. \square

In [15, Section 3] the authors established some essential properties of the class of deformations $\mathfrak{D}(\Omega, \Omega^*) := \mathfrak{D}^\rho(\Omega, \Omega^*)$ introduced in Definition 2.2, for $\rho \equiv 1$. Two main properties are that the family $\mathfrak{D}^\rho(\Omega, \Omega^*)$ is sequentially weakly closed (Lemma 2.12) and its members satisfy a change of variable formula (2.1). It must be noticed that all propositions of [15, Section 3] except [15, Lemma 3.13] can be extend as well to the class $\mathfrak{D}^\rho(\Omega, \Omega^*)$, with some nonessential changes that depends on the metric ρ ; for instance instead of $\int_{\Omega} J_h \leq |\Omega^*|$ use the inequality $\int_{\Omega} \rho^2(h(z)) J_h \leq \mathcal{A}(\rho)$ and instead of δ_f consider δ_f^ρ . Notice also that the degree theory [27] and Lusin's condition for the class $W^{1,2}$ [28] play important role in the proofs. We will only formulate the corresponding properties needed in this paper.

Lemma 2.4. [15, Lemma 3.2] *Let Ω , Ω^* and Ω° be domains in \mathbb{C} . If $f: \Omega^\circ \xrightarrow{\text{ontq}} \Omega$ is a quasiconformal mapping then for any $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ we have $h \circ f \in \mathfrak{D}^\rho(\Omega^\circ, \Omega^*)$.*

Observe that in Lemma 2.4 Ω and Ω^* need not be bounded domains. This do not creates problems because $\mathcal{E}[h \circ f] \leq K\mathcal{E}[f]$, where K is the quasiconformality constant of h .

Lemma 2.5. [15, Lemma 3.4] *For any $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ we have $h(\Omega) \supset \Omega^*$.*

Definition 2.6. A continuous mapping $f: \mathbb{X} \rightarrow \mathbb{Y}$ between metric spaces \mathbb{X} and \mathbb{Y} is *monotone* if for each $y \in f(\mathbb{X})$ the set $f^{-1}(y)$ is compact and connected.

Proposition 2.7. [35, VIII.2.2] *If \mathbb{X} is compact and $f: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$ is monotone then $f^{-1}(C)$ is connected for every connected set $C \subset \mathbb{Y}$.*

Lemma 2.8. [15, Lemma 3.7] *Let Ω and Ω^* be doubly connected domains in \mathbb{C} , and $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$. Then h is monotone.*

Lemma 2.9. [15, Lemma 3.8] *Let Ω and Ω^* be domains in \mathbb{C} . If $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$, then h satisfies Lusin's condition (N) and $N_\Omega(h, w) = 1$ for almost every $w \in \Omega^*$. Also $J_h = 0$ almost everywhere in $\Omega \setminus h^{-1}(\Omega^*)$.*

Corollary 2.10. [15, Corollary 3.9] *Let Ω and Ω^* be domains in \mathbb{C} . If $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ and $v: \Omega^* \rightarrow [0, \infty)$ is measurable, then*

$$(2.1) \quad \int_{\Omega} v(h(z)) J_h(z) dz = \int_{\Omega^*} v(w) dw.$$

In general, a deformation may take a part of Ω into $\partial\Omega^*$. This is the subject of next lemma.

Lemma 2.11. [15, Lemma 3.10] *Suppose that $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ where Ω and Ω^* are doubly connected domains. Let $G = \{z \in \Omega: h(z) \in \Omega^*\}$. Then G is a domain separating the boundary components of Ω . Precisely, the two components of $\partial\Omega$ lie in different components of $\mathbb{C} \setminus G$.*

Now we formulate the following extension of [15, Lemma 3.13].

Lemma 2.12. *Let Ω and Ω^* be bounded doubly connected planar domains. Assume that the boundary components of Ω do not degenerate into points. If a sequence $\{h_j\} \subset \mathfrak{D}^\rho(\Omega, \Omega^*)$ converges weakly in $W^{1,2}$, then its limit belongs to $\mathfrak{D}^\rho(\Omega, \Omega^*)$*

Proof. The key result needed for the proof of [15, Lemma 3.13] was [15, Proposition 3.11]. Due to different metric, instead of [15, Proposition 3.11] we need here a corollary of following Courant-Lebesgue lemma:

Lemma 2.13. [17, Lemma 1.3.2]. *Let $\Omega = A(r, R)$ be a circular annulus. Let Ω^* be a bounded doubly connected domain with allowable metric ρ and with distance function $\text{dist}_\rho(\cdot, \cdot)$. Let $0 < \epsilon < \min\{1, (R - r)/2\}$ and $z_0 \in \Omega$ and define $S(z_0, \epsilon) := \{z \in \Omega: |z - z_0| = \epsilon\}$. Suppose $f \in W^{1,2}(\Omega, \Omega^*)$, with $\mathcal{E}^\rho[f] \leq K$. Then there exists some $\nu: \epsilon < \nu < \sqrt{\epsilon}$ for which $f|_{S(z_0, \nu) \cap \Omega}$ is absolutely continuous and satisfies*

$$(2.2) \quad \text{dist}_\rho(f(z_1), f(z_2)) \leq (8\pi K)^{1/2} (\log(1/\epsilon))^{-1/2}$$

for all $z_1, z_2 \in S(z_0, \nu) \cap \Omega$.

Corollary 2.14. *Let $f \in \mathfrak{D}^\rho(\Omega, \Omega^*)$. Then for $\text{dist}(z, \partial\Omega) < \sqrt{3}\epsilon/2$ we have*

$$\text{dist}_\rho(f(z), \partial\Omega^*) \leq (8\pi \mathcal{E}^\rho[f])^{1/2} (\log(1/\epsilon))^{-1/2}.$$

Proof of Corollary 2.14. Let $K = \mathcal{E}^\rho[f]$. Pick $\epsilon > 0$ and let $z_0^n, n = 1, \dots, n_0$ be a sequence of points of Ω such that

$$\partial\Omega \subset \bigcup_{n=1}^{n_0} U(z_0^n, \nu_k/2) \text{ and } S(z_0^n, \epsilon) \cap \partial\Omega \neq \emptyset.$$

Here $\nu_k = \nu(z_0^n) \in [\epsilon, \sqrt{\epsilon}]$ is provided by Lemma 2.13 and by $U(z, r)$ we denote the open disk in the complex plane with the center z and the radius

r . It follows from (2.2) that

$$\text{dist}_\rho(f(z), \partial\Omega^*) \leq (8\pi K)^{1/2}(\log(1/\epsilon))^{-1/2}$$

provided that $z \in \Omega$ and $z \in S(z_0^n, \nu_k)$. Then for

$$z \in \partial\left(\bigcup_{n=1}^{n_0} U(z_0^n, \nu_k)\right) \cap \Omega$$

we have

$$\text{dist}_\rho(f(z), \partial\Omega^*) \leq (8\pi K)^{1/2}(\log(1/\epsilon))^{-1/2}.$$

Since, by Lemma 2.8 f is a monotone mapping, it follows that

$$\text{dist}_\rho(f(z), \partial\Omega^*) \leq (8\pi K)^{1/2}(\log(1/\epsilon))^{-1/2}$$

for $\text{dist}(z, \partial\Omega) < \sqrt{3}\epsilon/2$. \square

Let q be a conformal mapping of the annulus $A(\tau)$ onto Ω and p a conformal mapping of Ω^* onto $A(\omega)$ ($0 < \tau, \omega < \infty$). Then $f \in D^\rho(\Omega, \Omega^*)$ if and only if $h = p \circ f \circ q \in D^\rho(A(\tau), A(\omega))$. The rest of the proof of Lemma 2.12 is similar to that of [15, Lemma 3.13]. \square

Since the ρ -Dirichlet energy is weak lower semicontinuous (see for example [34, Lemma 2.1]), Lemma 2.1 has the following useful corollary.

Corollary 2.15. Under the hypotheses of Lemma 2.12 there exists $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ such that $\mathcal{E}^\rho[h] = \mathbf{E}^\rho(\Omega, \Omega^*)$.

Finally let us formulate the following property of Sobolev homeomorphisms.

Lemma 2.16. [15, Lemma 3.15] *Let Ω and Ω^* be doubly connected domains in \mathbb{C} . Then $H^{1,2}(\Omega, \Omega^*)$ is nonempty, except for one degenerate case when $\text{Mod } \Omega = \infty$ and $\text{Mod } \Omega^* < \infty$. In this case there is no homeomorphism $h: \Omega \xrightarrow{\text{onto}} \Omega^*$ of Sobolev class $W^{1,2}$.*

3. HARMONIC REPLACEMENT

Let Ω be a domain in \mathbb{C} and $U \Subset \Omega$ a simply connected domain. For a continuous function $f: \Omega \rightarrow \mathbb{C}$ the continuous function $\mathcal{H}_U f: \Omega \rightarrow \mathbb{C}$ is called the ρ -harmonic modification of f , if $\mathcal{H}_U f$ is harmonic in U and agrees with f on $\Omega \setminus U$. In order to apply "harmonic modification", in this section we made a small extension of classical Rado-Kneser-Choquet theorem for the functions which are merely monotone on the boundary (Lemma 3.2).

Definition 3.1. We say that a domain $K \Subset \Omega^*$ with Lipschitz boundary is allowable if K is convex w.r. to the metric ρ and is contained in a geodesic disk $B_\nu(p)$ with:

- (1) radius $\nu < \pi/(2\kappa)$;
- (2) the cut locus of the center p disjoint from $B_{2\nu}(p)$,

Lemma 3.2. *Let U be a simply connected domains in \mathbb{C} and D is an allowable domain in Ω^* . Suppose that f is a homeomorphism from U onto D with continuous extension $f: \bar{U} \rightarrow \bar{D}$. Then there exists a unique ρ -harmonic diffeomorphism $h: U \xrightarrow{\text{ontq}} D$ which agrees with f on the boundary. In particular, h has a continuous extension to \bar{U} which coincides with f on ∂U .*

Proof. Since a composition of a ρ -harmonic mapping with a conformal mapping is itself ρ -harmonic, by composing by a conformal mapping of the unit disk onto U we can assume that U is the unit disk. Assume for a moment that $f: \partial U \rightarrow \partial D$ is a homeomorphism. Consider the Dirichlet problem of finding a ρ -harmonic map $h: U \rightarrow \Omega^*$ with the given boundary values: $h|_{\partial U} = f$. By a result of Hildebrandt, Kaul and Widman [10] this Dirichlet problem has a solution contained in $B_\nu(p)$. Moreover by a result of Jost [18] we obtain that, since $h: \partial U \rightarrow \Omega^*$ is a homeomorphism onto a Lipschitz convex curve ∂D , then the above solution h is a homeomorphism.

Assume now that $f: \partial U \rightarrow \partial D$ is not a homeomorphism. Since ∂D is Lipschitz, it is a rectifiable curve with the length l . Let $\gamma: [0, l] \rightarrow \partial D$ be its arc-length parametrization and define $\tilde{f}: [0, 2\pi] \rightarrow [0, l]$ such that $\gamma(\tilde{f}(t)) = f(e^{it})$. By assumptions of the lemma, we can conclude that \tilde{f} is monotone. By using mollifiers we can define a sequence of C^2 diffeomorphisms $\tilde{f}_n: [0, 2\pi] \rightarrow [0, l]$ converging uniformly to \tilde{f} (see [9, p2. 351–352] for an explicit construction of the sequence \tilde{f}_n). Moreover we can assume that $\tilde{f}_n(a_k) = \tilde{f}(a_k)$, where $a_k \in [0, 2\pi)$, $k = 1, 2, 3$ are three different points. Let h_n be a ρ -harmonic diffeomorphism satisfying the boundary condition $h_n(e^{it}) = \gamma(\tilde{f}_n(t))$.

By [16, Corollary 7.1], on each disc $U(0, r) := \{z \in U: |z| < r\}$, $r < 1$, there is an a priori bound of the Jacobian determinant of $h_n(z)$ from below i.e.

$$(3.1) \quad |J_{h_n}(z)| \geq 1/\delta,$$

where $\delta = \delta(\kappa, \nu, r, \mathcal{E}^\rho[h|_U], |B_\nu(p)|)$ (δ also depends on a three point condition of h_n but this is satisfied because of the previous consideration). The class of functions h_n is equicontinuous on the closed unit disk \bar{U} (see [18, Lemma 4] for this argument). By Arzela-Ascoli theorem we can find a subsequence of h_n converging uniformly to a mapping h . Moreover, by virtue of [18, Lemma 5 a)], the derivatives of the sequence h_n up to the second order converge uniformly in compacts to the derivatives of the mapping h . Consequently the limit function h is of class C^2 and satisfies (3.1). This means that the Jacobian does not vanish. Therefore $h \in C^2(U) \cap C(\bar{U})$ is a ρ -harmonic function, has the prescribed boundary data f and its Jacobian is not vanishing in the interior of U . Therefore h is a local diffeomorphic proper mapping and by Banach-Mozur theorem h is a diffeomorphism in U . \square

We now apply the harmonic modification to deformations.

Lemma 3.3 (Modification Lemma). *Let Ω and Ω^* be doubly connected domains and assume that ρ is an allowable metric. Suppose that $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ satisfies $h(\Omega) = \Omega^*$. Let D be an allowable set such that $\overline{D} \subset \Omega^*$. Denote $U = h^{-1}(D)$. Then there exists $g = \mathcal{H}_U h$ and it satisfies the following properties*

- (a) $g \in \mathfrak{D}^\rho(\Omega, \Omega^*)$
- (b) *The restriction of g to U is a harmonic diffeomorphism onto D .*
- (c) $\mathcal{E}^\rho[g] \leq \mathcal{E}^\rho[h]$ *with equality if and only if $g \equiv h$.*

Proof. The proof is the same as the corresponding result in [15], but instead of [15, Lemma 4.1] we make use of Modification Lemma 3.2 to h . Moreover the inequality $\mathcal{E}^\rho[\mathcal{H}_U h] \leq \mathcal{E}^\rho[h]$ follows from the Dirichlet's principle applied to ρ -harmonic mapping h . \square

4. REICH-WALCZAK-TYPE INEQUALITIES REVISITED

Here we formulate two important propositions proved in [15]. Moreover we improve one of them. It must be said that, the inequalities do not depend on given metrics on domains. The inequalities in question are concerned with doubly connected domains. Similar inequalities were established in [29] in the context of self-homeomorphisms of a disk that agree with the identity mapping on the boundary.

By following [15] we introduce notation for several quantities associated with the derivatives of a mapping f . We use polar coordinates r and θ and the *normal* and *tangential* derivatives

$$f_N = f_r \quad \text{and} \quad f_T = \frac{f_\theta}{r}.$$

In these terms the complex partial derivatives f_z and $f_{\bar{z}}$ can be expressed as

$$f_z = \frac{e^{-i\theta}}{2} (f_N - i f_T) \quad \text{and} \quad f_{\bar{z}} = \frac{e^{i\theta}}{2} (f_N + i f_T).$$

The Jacobian determinant of f is

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = \operatorname{Im} \overline{f_N} f_T.$$

The *normal* and *tangential distortion* of f are defined as follows.

$$(4.1) \quad K_N^f := \frac{|f_z + \frac{\bar{z}}{z} f_{\bar{z}}|^2}{J_f} = \frac{|f_N|^2}{J_f}$$

$$(4.2) \quad K_T^f := \frac{|f_z - \frac{\bar{z}}{z} f_{\bar{z}}|^2}{J_f} = \frac{|f_T|^2}{J_f}$$

By convention we put $K_N^f = 0$ and $K_T^f = 0$ if the numerator vanishes and $K_N^f = \infty$ and $K_T^f = \infty$ if the J_f vanishes but the numerator does not. For a mapping $f \in W_{\text{loc}}^{1,1}$ the quantities f_N , f_T , and J_f are finite a.e. Thus K_N^f and K_T^f are defined a.e. on the domain of definition of f .

Proposition 4.1. [15, Proposition 5.1] *Let Ω and Ω^* be doubly connected domains such that Ω separates 0 and ∞ . Suppose that either*

(a) $f \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ or

(b) $f: \Omega \xrightarrow{\text{ont}\Omega} \Omega^*$ is a sense-preserving homeomorphism of class $W_{\text{loc}}^{1,1}(\Omega, \Omega^*)$.

Then

$$(4.3) \quad 2\pi \text{Mod } \Omega^* \leq \int_{\Omega} K_N^f \frac{dz}{|z|^2}.$$

Unlike [15, Proposition 5.2], our lower bound for the modulus of the image under a deformation do not depends on the rectifiability of the boundary of Ω^* . This implies that the assumption of rectifiability in [15, Proposition 5.2] is redundant. Namely we have:

Proposition 4.2. *Let $\mathbb{A} = A(r, R)$ be a circular annulus, $0 \leq r < R < \infty$, and Ω^* a doubly connected domain with finite modulus and suppose that either*

a) $f \in \mathfrak{D}^\rho(\mathbb{A}, \Omega^*)$ or

b) $f: \mathbb{A} \xrightarrow{\text{ont}\Omega} \Omega^*$ is a sense-preserving homeomorphism of class $W_{\text{loc}}^{1,1}(\mathbb{A}, \Omega^*)$.

Then

$$(4.4) \quad \int_{\mathbb{A}} K_T^f \frac{dz}{|z|^2} \geq 2\pi \frac{(\text{Mod } \mathbb{A})^2}{\text{Mod } \Omega^*}.$$

Proof. The item b) is proved in [15, Proposition 5.2]. Prove the item a). Let Φ be a conformal mapping of Ω^* onto the annulus $A(r^*, 1)$. Then

$$\text{Mod}(\Omega^*) = \text{Mod}(A(r^*, 1)) = \log \frac{1}{r^*}.$$

Further for $F = \Phi \circ f$ we have

$$J_F(z) = J_\Phi(f(z)) \cdot J_f(z) = |\phi'(f(z))|^2 J_f(z)$$

and

$$F_T = \Phi'(f(z)) f_T.$$

Therefore

$$K_T^F = \frac{|F_T(z)|^2}{J_F(z)} = \frac{|f_T(z)|^2}{J_f(z)} = K_T^f.$$

From Lemma 2.4, $F \in \mathfrak{D}(\mathbb{A}, A(r^*, 1))$. We conclude the proof by invoking [15, Proposition 5.2. a)] to the mapping F and annuli \mathbb{A} and $A(r^*, 1)$ which has a rectifiable boundary. \square

5. STATIONARY DEFORMATIONS

We call a deformation $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ *stationary* if

$$(5.1) \quad \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}^\rho[h \circ \phi_t^{-1}] = 0$$

for every family of diffeomorphisms $t \rightarrow \phi_t: \Omega \rightarrow \Omega$ which depend smoothly on the parameter $t \in \mathbb{R}$ and satisfy $\phi_0 = \text{id}$. The latter mean that the

mapping $\Omega \times [0, \epsilon_0] \ni (t, z) \rightarrow \phi_t(z) \in \Omega$ is a smooth mapping for some $\epsilon_0 > 0$. Assume as we may that Ω is a doubly connected domain with smooth boundary. The derivative in (5.1) exists for any $h \in W^{1,2}(\Omega)$, see computation in [17, p. 153-154] (c.f. [34, p. 158]). Every energy-minimal deformation is stationary. Indeed, $h \circ \phi_t^{-1}$ belongs to $\mathfrak{D}^\rho(\Omega, \Omega^*)$ if t is close to zero by virtue of Lemma 2.4, because $\phi_t = \text{id} + o(t)$. The minimal property of h implies $\mathcal{E}^\rho[h \circ \phi_t^{-1}] \geq \mathcal{E}^\rho[h]$, from where we obtain (5.1).

Following verbatim the proof of [17, Lemma 1.2.2] but beginning by

$$(5.2) \quad \varphi(z) dz^2 := \text{Hopf}(h) = \frac{\rho^2(h(z))}{4} (|h_x|^2 - |h_y|^2 - 2i \langle h_x, h_y \rangle) dz^2$$

instead of [17, Eq. (1.2.24)] we obtain the following crucial properties of the stationary mapping in (5.1):

- The function $\varphi := \rho^2(h(z)) h_z \overline{h_z}$, a priori in $L^1(\Omega)$, is holomorphic.
- If $\partial\Omega$ is C^1 -smooth then φ extends continuously to $\overline{\Omega}$, and the quadratic differential φdz^2 is real on each boundary curve of Ω .

Let us consider the particular case $\Omega = A(r, R)$ with $0 < r < R < \infty$. Since φdz^2 is real on each boundary circle: $z = pe^{it}$, ($p = r, R$), the differential $\varphi dz^2 = -\varphi(z) p^2 e^{2it} dt^2 = -z^2 \varphi(z) dt^2$ is real on $\partial\Omega$. Thus the function $z^2 \varphi(z)$ is real on $\partial\Omega$, and by the maximum principle to the harmonic function $\text{Im}(z^2 \varphi(z))$ it follows that

$$(5.3) \quad z^2 \varphi(z) \equiv c \in \mathbb{R}.$$

We now have.

Lemma 5.1. *Let $\Omega = A(r, R)$ be a circular annulus, $0 < r < R < \infty$, and Ω^* a doubly connected domain. If $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ is a stationary deformation, then*

$$(5.4) \quad \rho^2(h(z)) h_z \overline{h_z} \equiv \frac{c}{z^2} \quad \text{in } \Omega$$

where $c \in \mathbb{R}$ is a constant. Furthermore,

$$(5.5) \quad \begin{cases} |h_N|^2 \leq J_h, & \text{if } c \leq 0 \\ |h_T|^2 \leq J_h, & \text{if } c \geq 0. \end{cases}$$

Finally if ρ is bounded, then

$$(5.6) \quad |Dh|^2 \geq \frac{4|c|}{R^2 \rho_0^2},$$

where $\rho_0 = \sup_{w \in \Omega^*} \rho(w)$.

Proof. Proof goes along the lines of the proof of [15, Lemma 6.1] but since we need some relations of the proof we include the proof here. The relation (5.4) with some $c \in \mathbb{R}$ was already established in (5.3). Separating the real and

imaginary parts in (5.4) we arrive at two equations

$$(5.7) \quad \rho^2(h(z))(|h_N|^2 - |h_T|^2) = \frac{4c}{|z|^2};$$

$$(5.8) \quad \rho^2(h(z)) \operatorname{Re}(\overline{h_N} h_T) = 0.$$

Recall that $J_h = \operatorname{Im} \overline{h_N} h_T \geq 0$, which in view of (5.8) reads as

$$(5.9) \quad J_h = |h_N| |h_T|.$$

Combining this with (5.7) the claim (5.5) follows. The relation (5.6) follows from (5.4) and the fact that $|h_z| \geq |h_{\bar{z}}|$. \square

Lemma 5.1 together with Propositions 4.1 and 4.2 give the following improvement of [15, Corollary 6.2].

Corollary 5.2. Under the hypotheses of Lemma 5.1, we have

- if $\operatorname{Mod} \Omega < \operatorname{Mod} \Omega^*$, then $c > 0$
- if $\operatorname{Mod} \Omega > \operatorname{Mod} \Omega^*$, then $c < 0$.

6. MONOTONICITY OF MINIMUM ENERGY FUNCTION

Because the Dirichlet integral and the class of deformations are conformally invariant (Lemma 2.4), the minimal energy $E^\rho(\Omega, \Omega^*)$, defined by (1.10), depends only on the conformal type of Ω provided that Ω^* is fixed. Moreover by Lemma 2.3, we can assume that Ω^* is a circular annulus $A(r, 1)$. This leads us to consider a one-parameter family of extremal problems for homeomorphisms $A(\tau) \xrightarrow{\text{ontg}} A(\omega)$. In this section consider the quantity $E^\rho(\tau, \omega) := E^\rho(A(\tau), A(\omega))$ as a function of τ , called the *minimum energy function*. Notice that $A(\tau)$ is conformally equivalent to $A(1, e^\tau)$ and the latter has been used in [15], however for some technical reasons, see Section 9, $A(\tau)$ has been shown to be more appropriate. It is clear that the function $E^\rho(\tau, \omega)$ attains its minimum at $\tau = \omega$. Indeed, by (1.6) for every τ we have $E^\rho(\tau, \omega) \geq 2\mathcal{A}(\rho)$, with equality if and only if $\tau = \omega$. The following monotonicity result, which extends this observation, will be very important in the proof of Theorem 1.4. It extends and improves the corresponding [15, Proposition 7.1].

Proposition 6.1. *Let $\omega > 0$ and ρ be a smooth metric with bounded Gauss curvature in $A(\omega)$. The function $\tau \mapsto E^\rho(\tau, \omega)$ is strictly decreasing for $0 < \tau < \omega$ and strictly increasing for $\tau > \omega$. Furthermore*

$$\left. \frac{d}{dt} \right|_{\tau=\tau_0} E^\rho(\tau, \omega) = -8\pi c,$$

where c is defined in Lemma 5.1 and the constant c depends only on τ and ω .

Similarly as in [15], the proof of Proposition 6.1 requires auxiliary results concerning the normal and tangential energies

$$\mathcal{E}_N^\rho[h] = \int_{\Omega} \rho^2 |h_N|^2, \quad \mathcal{E}_T^\rho[h] = \int_{\Omega} \rho^2 |h_T|^2.$$

First of all $\mathcal{E}^\rho[h] = \mathcal{E}_N^\rho[h] + \mathcal{E}_T^\rho[h]$. Both functionals $\mathcal{E}_N^\rho[h]$ and $\mathcal{E}_T^\rho[h]$ transform in a straightforward way under composition with the power stretch mapping

$$(6.1) \quad \psi(z) := |z|^{\alpha-1} z, \quad 0 < \alpha < \infty.$$

By using the formula $\det(D\psi(z)) = \alpha |z|^{2\alpha-2}$, we obtain

$$(6.2) \quad \mathcal{E}_N^\rho[h \circ \psi] = \alpha \mathcal{E}_N^\rho[h], \quad \mathcal{E}_T^\rho[h \circ \psi] = \frac{1}{\alpha} \mathcal{E}_T^\rho[h].$$

As in [15], the domain of definition of h here is irrelevant because the computation is local.

Lemma 6.2. *Let $\omega \in (0, \infty)$ and $\tau_o \in (0, \infty)$. Suppose that $h^\circ \in \mathfrak{D}^\rho(A(\tau_o), A(\omega))$ is an energy-minimal deformation. Then for all $0 < \tau < \infty$ we have*

$$(6.3) \quad \mathbf{E}^\rho(\tau, \omega) \leq \frac{\tau_o}{\tau} \mathcal{E}_N^\rho[h^\circ] + \frac{\tau}{\tau_o} \mathcal{E}_T^\rho[h^\circ].$$

Proof. Proof goes along the lines of the proof of [15, Lemma 7.2]. \square

Let us apply Lemma 6.2 with $\tau_o = \omega$. In this case $h^\circ: \Omega \xrightarrow{\text{ontq}} \Omega^*$ is conformal so $\mathcal{E}_N^\rho[h^\circ] = \mathcal{E}_T^\rho[h^\circ] = \mathcal{A}(\rho)$. We obtain the following simple upper bound for the minimal energy function,

$$(6.4) \quad \mathbf{E}^\rho(\tau, \omega) \leq \left(\frac{\omega}{\tau} + \frac{\tau}{\omega} \right) \mathcal{A}(\rho), \quad 0 < \tau < \infty.$$

Corollary 6.3. The function $\mathbf{E}^\rho(\tau, \omega)$ is locally Lipschitz for $0 < \tau < \infty$.

Indeed the existence of h° in Lemma 6.2 is assured by Corollary 2.15. From Lemma 6.2 for arbitrary $0 < \tau_o, \tau < \infty$ we have

$$(6.5) \quad \begin{aligned} \mathbf{E}^\rho(\tau, \omega) - \mathbf{E}^\rho(\tau_o, \omega) &\leq \frac{\tau_o}{\tau} \mathcal{E}_N^\rho[h^\circ] + \frac{\tau}{\tau_o} \mathcal{E}_T^\rho[h^\circ] - \mathcal{E}_N^\rho[h^\circ] - \mathcal{E}_T^\rho[h^\circ] \\ &= (\tau - \tau_o) \left\{ \frac{\mathcal{E}_T^\rho[h^\circ]}{\tau_o} - \frac{\mathcal{E}_N^\rho[h^\circ]}{\tau} \right\} \end{aligned}$$

From this we obtain the local Lipschitz property.

Proof of Proposition 6.1. Since $\mathbf{E}^\rho(\tau, \omega)$ is locally Lipschitz, its derivative exists for almost every $\tau \in (0, \infty)$. Fix such a point of differentiability, say $0 < \tau_o < \text{Mod } \Omega^*$ (the other possibility is $\tau_o > \text{Mod } \Omega^*$). Then as in [15, Proposition 7.2] we obtain

$$\left. \frac{d}{dt} \right|_{\tau=\tau_o} \mathbf{E}^\rho(\tau, \omega) = -8\pi c.$$

Corollary 5.2 completes the proof. This shows that c depends only on τ_o, ω and ρ but not on h° . \square

7. PROOF OF THEOREM 1.4 AND THEOREM 1.3

In order to prove our main result we need the following extension of [15, Proposition 8.1].

Proposition 7.1. *Let Ω and Ω^* be doubly connected domains. Suppose that $h \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ satisfies $\mathcal{E}^\rho[h] = E^\rho(\Omega, \Omega^*)$. Let $G = \{z \in \Omega : h(z) \in \Omega^*\}$. Then G is a doubly connected domain which separates the boundary components of Ω . The restriction of h to G is a ρ -harmonic diffeomorphism onto Ω^* .*

Proof. The fact that G is a domain that separates the boundary components of Ω was established in Lemma 2.11. For every point $z \in G$ there exists a neighborhood $D = D_x$ in which h is a harmonic diffeomorphism. Indeed, otherwise we would be able to find a ρ deformation with strictly smaller energy by means of Lemma 3.3. Namely for $w = h(z) \in \Omega^*$ we can find an allowable neighborhood $D \subset \Omega^*$. Then by taking $U = h^{-1}(D)$ and making use of Lemma 3.3 for G instead of Ω we obtain the previous fact. Thus, $h : G \xrightarrow{\text{onto}} \Omega^*$ is a local homeomorphism. By an extension of Lewy theorem due to Heinz, [8, Theorem 11], h is a local diffeomorphism. On the other hand, for each $w \in \Omega^*$ the preimage $h^{-1}(w)$ is connected by Lemma 2.8. It follows that $h : G \xrightarrow{\text{onto}} \Omega^*$ is a diffeomorphism. Being a diffeomorphic image of Ω^* , the domain G must be doubly connected. \square

Proof of Theorem 1.4. If $\text{Mod } \Omega = \text{Mod } \Omega^*$, then the domains are conformally equivalent. As observed in §1, a conformal mapping minimizes the ρ -Dirichlet energy. Thus we only need to consider the case $\text{Mod } \Omega < \text{Mod } \Omega^*$. Assume that $a : \Omega \xrightarrow{\text{onto}} A(\tau)$ and $b : A(\omega) \xrightarrow{\text{onto}} \Omega^*$ are conformal mappings.

Let h and G be as in Proposition 7.1. The existence of such h is guaranteed by Corollary 2.15. Since G separates the boundary components of Ω , we have $\text{Mod } G \leq \text{Mod } A(\tau) = \tau$ with equality if and only if $G = \Omega$ [26, Lemma 6.3]. If $\text{Mod } G < \tau$, then by Proposition 6.1

$$\int_G \rho^2 \circ h |Dh|^2 \geq E^\rho(\text{Mod } G, \Omega^*) > E^\rho(\tau, \Omega^*) = \int_{A(\tau)} \rho^2 \circ h |Dh|^2$$

which is absurd because $G \subset A(\tau)$. Thus $G = A(\tau)$. By Proposition 7.1 the mapping $h : A(\tau) \rightarrow A(\omega)$ is a harmonic diffeomorphism. The uniqueness statement will follow from Proposition 8.2. Then, by Lemma 2.3 h is a minimizer of $\rho_1 := \rho \circ b \cdot |b'|$ -energy between $A(\tau)$ and $A(\omega)$. Let $f^\circ = b \circ h \circ a$. Then $f^\circ : \Omega \xrightarrow{\text{onto}} \Omega^*$ is a ρ -harmonic mapping. Moreover by Lemma 2.3, f° is a minimizer because h is a minimizer. Indeed by Lemma 2.3 $g \in \mathfrak{D}^\rho(A(\tau), A(\omega))$ if and only if $f = b \circ g \circ a \in \mathfrak{D}^\rho(\Omega, \Omega^*)$ and $\mathcal{E}^\rho[b \circ g \circ a] = \mathcal{E}^{\rho_1}[g]$.

Thus

$$\begin{aligned}\mathcal{E}^{\rho_1}[h] &= \min\{\mathcal{E}^{\rho_1}[g] : g \in \mathfrak{D}^\rho(A(\tau), A(\omega))\} \\ &= \min\{\mathcal{E}^\rho[b \circ g \circ a] : g \in \mathfrak{D}^\rho(A(\tau), A(\omega))\} \\ &= \min\{\mathcal{E}^\rho[f] : f \in \mathfrak{D}^\rho(\Omega, \Omega^*)\} = \mathcal{E}^\rho[f^\circ].\end{aligned}$$

□

Proof of Theorem 1.3. Suppose $\text{Mod } \Omega \leq \text{Mod } \Omega^*$ and let $f_\circ : \Omega \xrightarrow{\text{ontq}} \Omega^*$ be an energy-minimal diffeomorphism provided to us by Theorem 1.4. Assume as we may also that $\Omega = A(r, 1)$. For every homeomorphism $g : \Omega^* \xrightarrow{\text{ontq}} \Omega$ with $L^1 = L^1(\Omega^*, d\mu)$ integrable distortion the inverse map $f = g^{-1} : \Omega \xrightarrow{\text{ontq}} \Omega^*$ belongs to the Sobolev class $W_{loc}^{1,2}(\Omega)$ (because ρ is smooth in Ω). Let $\Omega_n = A(r + 1/n, 1 - 1/n)$ be an exhaustion by compact sets of $\Omega = A(r, 1)$ and let $\Omega_n^* = f(\Omega_n)$. Then from Corollary 8.5 below and [6] we have

$$\begin{aligned}(7.1) \quad \int_{\Omega^*} \rho^2(w) K_g(w) dw &= \lim_{n \rightarrow \infty} \int_{\Omega_n^*} \rho^2(w) K_g(w) dw \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_n} \rho^2(f(z)) |Df(z)|^2 dz \\ &\geq \lim_{n \rightarrow \infty} \int_{\Omega_n} \rho^2(f_\circ(z)) |Df_\circ(z)|^2 dz \\ &= \int_{\Omega} \rho^2(f(z)) |Df_\circ(z)|^2 dz \\ &= \int_{\Omega^*} \rho^2(w) K_{g_\circ}(w) dw\end{aligned}$$

where $g_\circ = f_\circ^{-1}$. The latter follows by change of variables in integral, where the substitution function is the C^∞ -smooth diffeomorphism g_\circ . If equality holds in (7.1) then, by Theorem 1.4, the mapping $f_\circ^{-1} \circ f$ is conformal. □

8. CONVEXITY OF THE MINIMUM ENERGY FUNCTION

In §6 we proved that for every doubly connected domain Ω^* the function $E^\rho(\tau, \omega)$ is decreasing for $0 < \tau < \omega$ and increasing for $\tau > \omega$. The minimum of this function is attained at $\tau = \omega$, i.e., in the case of conformal equivalence. In this section we prove:

Theorem 8.1. *Let $\omega > 0$. The function $\tau \mapsto E^\rho(\tau, \omega)$ is strictly convex for $0 < \tau < \omega$.*

Theorem 8.1 is an extension of corresponding [15, Theorem 10.1], which is, according to Lemma 2.3 a special case of Theorem 8.1 for ρ being a modulus of an analytic function. Before proving we establish some propositions which we believe can be of interest in its own right. On the other hand the item (2) of Proposition 8.2 establishes the uniqueness part of Theorem 1.1.

Proposition 8.2. *Let $\omega > 0$. Suppose that $h \in \mathfrak{D}^\rho(A(\tau), A(\omega))$ and $h^\circ \in \mathfrak{D}^\rho(A(\tau_\circ), A(\omega))$ are energy-minimal deformations. In particular, by Lemma 5.1,*

$$(8.1) \quad \rho^2(h(z))h_z\overline{h_z} \equiv \frac{c}{z^2} \quad \text{in } A(\tau)$$

and

$$(8.2) \quad \rho^2(h^\circ(z))h_z^\circ\overline{h_z^\circ} \equiv \frac{c_\circ}{z^2} \quad \text{in } A(\tau_\circ).$$

Let $G^\circ = (h^\circ)^{-1}(A(\omega))$ and $G = h^{-1}(A(\omega))$. Then

$$(8.3) \quad \mathcal{E}^\rho[h^\circ] - \mathcal{E}^\rho[h] \geq 4|c_\circ| \int_{A(\tau_\circ) \setminus G^\circ} \frac{dz}{|z|^2} + 8\pi c(\tau - \text{Mod}(G^\circ))$$

and

$$(8.4) \quad \mathcal{E}^\rho[h] - \mathcal{E}^\rho[h^\circ] \geq 4|c| \int_{A(\tau) \setminus G} \frac{dz}{|z|^2} + 8\pi c_\circ(\tau_\circ - \text{Mod}(G)).$$

Moreover:

(1) Equalities hold in both (8.3) and (8.4) if and only if G° and G are circular annuli and $(h^\circ)^{-1} \circ h$ is a conformal mapping between G and G° .

(2) If h° and h are minimizers of \mathcal{E}^ρ over the domain $A(\tau)$, then

$$h|_{G^\circ} = h^\circ|_G \circ f,$$

where $f: G \xrightarrow{\text{ontq}} G^\circ$ is a conformal mapping and

$$\int_G \frac{dz}{|z|^2} = \int_{G^\circ} \frac{dz}{|z|^2}.$$

In particular $\text{Mod}(G) = \text{Mod}(G^\circ)$.

Proof. We will only sketch the proof of (8.1) ((8.2)), since the corresponding proof of [15, Proposition 10.2] applies for our proof. First we consider the easy case $c = 0$. In this case h is conformal, which implies $\mathcal{E}^\rho[h] = 2\mathcal{A}(\rho)$. On the other hand, $\mathcal{E}^\rho[h^\circ] \geq 2\mathcal{A}(\rho)$ with equality if and only if h° is conformal, see (1.6).

Let $G := \{z \in A(\tau): h(z) \in A(\omega)\}$ and $G^\circ := \{z \in A(\tau_\circ): h^\circ(z) \in A(\omega)\}$. Then, by Proposition 7.1 G and G° are doubly connected domains that separates the boundary components of $A(\tau)$ and $A(\tau_\circ)$ respectively. Moreover

$$h^\circ: G^\circ \xrightarrow{\text{ontq}} A(\omega) \quad \text{and} \quad h: G \xrightarrow{\text{ontq}} A(\omega).$$

It remains to deal with $c \neq 0$. The composition

$$f = (h^\circ)^{-1} \circ h: A(\tau) \xrightarrow{\text{ontq}} G^\circ$$

lies in $W_{\text{loc}}^{1,2}(A(\tau))$ and is not homotopic to a constant mapping. Moreover, the restriction of f to the domain G is a diffeomorphism onto G° , by virtue of Proposition 7.1. Thus, f possesses a right inverse $f^{-1}: G^\circ \xrightarrow{\text{ontq}} G$ which is also a diffeomorphism. Now we estimate $\mathcal{E}^\rho[h^\circ] - \mathcal{E}^\rho[h]$ by using the change of variables $w = f(z)$ and proceeding similarly as in [15, Proposition 10.2],

but using this time our improved Proposition 4.2 (b) and Lemma 2.9. Let us only state one of key sequences of relations for the proof.

$$\begin{aligned}
(8.5) \quad & \mathcal{E}^\rho[h^\circ] - 4|c_\circ| \int_{A(\tau_\circ) \setminus G^\circ} \frac{dz}{|z|^2} - \int_G \rho^2(h(z)) |Dh|^2 \\
&= 4 \int_G \rho^2(h(z)) \frac{(|h_z|^2 + |h_{\bar{z}}|^2) |f_{\bar{z}}|^2 - 2 \operatorname{Re} [h_z \overline{h_{\bar{z}}} f_z f_{\bar{z}}]}{J_f} dz \\
&\geq 4 \int_G \rho^2(h(z)) \frac{2|h_z h_{\bar{z}}| |f_{\bar{z}}|^2 - 2 \operatorname{Re} [h_z \overline{h_{\bar{z}}} f_z f_{\bar{z}}]}{J_f} dz \\
&= 4|c| \int_G \left[\frac{|f_z - \sigma f_{\bar{z}}|^2}{J_f} - 1 \right] \frac{dz}{|z|^2}, \quad \text{where } \sigma = \sigma(z) = \frac{c\bar{z}}{|c|z}.
\end{aligned}$$

Proof of statement (1). Since h is a sense-preserving diffeomorphism in G° , we have $|h_z| > |h_{\bar{z}}|$ everywhere in G° . If equality holds in (8.3), then it also holds in (8.5). The latter is only possible if $f_{\bar{z}} \equiv 0$ in G . Thus $f: G \xrightarrow{\text{onto}} G^\circ$ is a conformal mapping. This implies $\operatorname{Mod}(G) = \operatorname{Mod}(G^\circ)$. Moreover, by adding (8.3) and (8.4) we have

$$\int_{G^\circ} \frac{dz}{|z|^2} = 2\pi \operatorname{Mod}(G^\circ)$$

and

$$\int_G \frac{dz}{|z|^2} = 2\pi \operatorname{Mod}(G).$$

To continue we need the following lemma

Lemma 8.3. *For a doubly connected domain G separating 0 and ∞ we have the inequality*

$$(8.6) \quad \int_G \frac{dz}{|z|^2} \geq 2\pi \operatorname{Mod}(G).$$

The equality is attained if and only if G is a circular annulus.

Proof of Lemma 8.3. The inequality statement can be deduced for example from [15, Proposition 5.1]. In order to prove the equality statement we should include an inequalities of [15, Proposition 5.1] and study it more closely. One of inequalities in [15, Proposition 5.1] implying (8.6) is

$$(8.7) \quad \left(\int_G \frac{|a'|}{|a|} \frac{dz}{|z|} \right)^2 \leq \int_G \frac{|a'|^2}{|a|^2} \int_G \frac{dz}{|z|^2},$$

where a is a conformal mapping of G onto an annulus $A(r_*, 1)$. The equality is attained in Cauchy-Schwarz inequality (8.7) if and only if

$$\frac{|a'|}{|a|} = \frac{\alpha}{|z|}.$$

Thus $a(z) = \beta z$ for some constant $\beta \neq 0$, implying that G is a circular annulus. \square

From Lemma 8.3 we deduce that G and G° are circular annuli.

Proof of statement (2). Assume that $h^\circ \in \mathfrak{D}^\rho(A(\tau_\circ), A(\omega))$ and $h \in \mathfrak{D}^\rho(A(\tau_\circ), A(\omega))$ are the minimizers of the energy \mathcal{E}^ρ . Then $\mathcal{E}^\rho[h^\circ] = \mathcal{E}^\rho[h]$. Since c depends only on τ , it follows that $c_\circ = c$. Let G and G° and f be defined as above. Without loss of generality we can assume that

$$(8.8) \quad \int_G \frac{dz}{|z|^2} \geq \int_{G^\circ} \frac{dz}{|z|^2}.$$

From (8.5), we have

$$\begin{aligned} \mathcal{E}^\rho[h^\circ] - \mathcal{E}^\rho[h] - 4|c| \int_{A(\tau_\circ) \setminus G^\circ} \frac{dz}{|z|^2} + 4|c| \int_{A(\tau_\circ) \setminus G} \frac{dz}{|z|^2} \\ = 4 \int_G \rho^2(h(z)) \frac{(|h_z|^2 + |h_{\bar{z}}|^2) |f_{\bar{z}}|^2 - 2 \operatorname{Re} [h_z \overline{h_{\bar{z}}} f_z f_{\bar{z}}]}{J_f} dz. \end{aligned}$$

Assume that

$$\mathcal{E}^\rho[h^\circ] = \mathcal{E}^\rho[h].$$

Then

$$(8.9) \quad \begin{aligned} 4 \int_G \rho^2(h(z)) \frac{(|h_z|^2 + |h_{\bar{z}}|^2) |f_{\bar{z}}|^2 - 2 \operatorname{Re} [h_z \overline{h_{\bar{z}}} f_z f_{\bar{z}}]}{J_f} \\ + 8|c|\pi \left(\int_G \frac{dz}{|z|^2} - \int_{G^\circ} \frac{dz}{|z|^2} \right) = 0. \end{aligned}$$

This implies that $f_{\bar{z}} \equiv 0$, i.e. f is a conformal mapping and therefore $\operatorname{Mod}(G) = \operatorname{Mod}(G^\circ)$ and

$$\int_G \frac{dz}{|z|^2} - \int_{G^\circ} \frac{dz}{|z|^2} = 0.$$

This finishes the proof of Proposition 8.2. \square

By following the proof of Proposition 8.2 we obtain the following useful variation of [15, Proposition 10.2].

Proposition 8.4. *Let Ω^* be a doubly connected domain. Suppose that $h \in \mathfrak{D}^\rho(A(\tau_\circ), \Omega^*)$ is a diffeomorphic deformation with Hopf differential cdz^2/z^2 . Then for any diffeomorphism $g: A(\tau) \rightarrow \Omega^*$ we have*

$$(8.10) \quad \mathcal{E}^\rho[g] - \mathcal{E}^\rho[h] \geq 8\pi c(\tau_\circ - \tau).$$

The equality holds in (8.10) if and only if $\tau = \tau_\circ$ and $g^{-1} \circ h$ is a conformal mapping of $A(\tau_\circ)$ onto itself.

Now we can prove the following important corollary

Corollary 8.5. *If $h \in \mathfrak{D}^\rho(A(\tau_\circ), \Omega^*)$ is a diffeomorphic deformation with Hopf differential cdz^2/z^2 then h is a minimizer (under the class of deformation) and it is unique up to reparametrization by a conformal mapping. In particular the restriction of every such deformation in a circular annulus $A' \subset A(\tau_\circ)$ is a minimizer.*

Proof of Corollary 8.5. Assume that f is a deformation. Let f_j be a *approximating sequence* of diffeomorphisms $f_j: \Omega \xrightarrow{\text{ontg}} \Omega^*$ such that $f_j \xrightarrow{c\delta} f$ on Ω . Since \mathcal{E}^ρ is weak lower semicontinuous (see [34, Lemma 2.1]) it follows that

$$\limsup_{j \rightarrow \infty} \mathcal{E}^\rho[f_j] \leq E^\rho[f].$$

On the other hand from (8.3) by taking $\tau_\circ = \tau$ we have $\mathcal{E}^\rho[h] \leq \mathcal{E}^\rho[f_j]$. Therefore $\mathcal{E}^\rho[h] \leq \mathcal{E}^\rho[f]$. \square

Proof of Theorem 8.1. Proof goes along the lines of the proof of Lemma [15, Theorem 10.1] by using Proposition 8.4. \square

Corollary 8.6. Let $\mathcal{T} = \{\tau : E^\rho(\tau, \omega) = \mathcal{E}^\rho[g] \text{ for some } g \in H^{1,2}(\Omega, \Omega^*)\}$ and $\tau_\circ = \sup \mathcal{T}$. Then the function $c = c(\tau)$ is a strictly decreasing function in \mathcal{T} .

Proof. Suppose that Let $0 < \tau_\circ, \tau < \tau_\circ$ and $\tau_\circ, \tau \in \mathcal{T}$. Assume that $h \in \mathfrak{D}^\rho(A(\tau_\circ), \Omega^*)$ and $g \in \mathfrak{D}^\rho(A(\tau), \Omega^*)$ are diffeomorphic energy-minimal deformations. Assume that $\tau < \tau_\circ$. Prove that $c(\tau_\circ) < c(\tau)$. From (8.10) we have

$$\mathcal{E}^\rho[g] - \mathcal{E}^\rho[h] \geq 8\pi c(\tau_\circ)(\tau_\circ - \tau)$$

and

$$\mathcal{E}^\rho[h] - \mathcal{E}^\rho[g] \geq 8\pi c(\tau)(\tau - \tau_\circ).$$

Dividing by $\tau_\circ - \tau$ the previous inequalities we arrive at inequality $c(\tau_\circ) \leq c(\tau)$. Because $A(\tau_\circ)$ and $A(\tau)$ are not conformally equivalent it follows that $c(\tau_\circ) < c(\tau)$ as desired. \square

Remark 8.7. Let $\tau_n \in \{\tau : E^\rho(\tau, \omega) = \mathcal{E}^\rho[g] \text{ for some } g \in H^{1,2}(A(\tau), \omega)\}$ be a sequence converging to τ_\circ and let g_n be a sequence of harmonic diffeomorphisms such that $E^\rho(\tau_n, \omega) = \mathcal{E}^\rho[g_n]$. Then g_n , up to some subsequence, converges to a harmonic diffeomorphism $h_\circ \in H^{1,2}(A(\tau_\circ), \omega)$. It is called the critical ρ -Nitsche map and $A(\tau_\circ)$ is called the critical domain for Ω^* . The explicit evaluation of τ_\circ is not known for arbitrary Ω^* i.e. for arbitrary ρ . However for circular annuli and radial metrics (in particular for Euclidean metric) the constant τ_\circ is calculated exactly (see [2] and [22]). The strict convexity part of Theorem 8.1 fails for $\tau > \tau_\circ$. We demonstrate this with Theorem 9.4 based on the results of [22].

9. CONVEXITY OF MINIMAL ENERGY FOR RADIAL METRICS

Definition 9.1. The radial metric ρ is called a *regular metric* if

$$\inf_{\delta < s < \sigma} s\rho(s) = \lim_{s \rightarrow \delta+0} s\rho(s)$$

and has bounded Gauss curvature K .

From now on we will assume that the metric ρ is regular in the sense of Definition 9.1. We recall first some results from [22]. Namely in [22]

are found all examples w of radial ρ -harmonic maps between annuli. The mapping w given by

$$(9.1) \quad w(se^{it}) = q^{-1}(s)e^{it},$$

where

$$(9.2) \quad q(s) = \exp \left(\int_{\sigma}^s \frac{dy}{\sqrt{y^2 + \gamma \varrho^2}} \right), \quad \delta \leq s \leq \sigma,$$

and γ satisfies the condition:

$$(9.3) \quad y^2 + \gamma \varrho^2(y) \geq 0, \text{ for } \delta \leq s \leq \sigma,$$

is a ρ -harmonic mapping between annuli $A = A(r, 1)$ and $A' = A(\delta, \sigma)$, where

$$(9.4) \quad r = \exp \left(\int_{\sigma}^{\delta} \frac{dy}{\sqrt{y^2 + \gamma \varrho^2}} \right).$$

The harmonic mapping w is normalized by

$$w(e^{it}) = \sigma e^{it}.$$

The mapping $w = h^{\gamma}(z)$ is a diffeomorphism, and is called ρ -Nitsche map.

Then (9.3) is equivalent to

$$(9.5) \quad \delta^2 + \gamma \varrho^2(\delta) \geq 0.$$

Accordingly, for $\gamma = -\delta^2 \rho^2(\delta)$, we have well defined function

$$(9.6) \quad q_{\diamond}(s) = \exp \left(\int_{\sigma}^s \frac{dy}{\sqrt{y^2 - \delta^2 \rho^2(\delta) \varrho^2}} \right), \quad \delta \leq s \leq \sigma.$$

The mapping $h_{\diamond} : A \rightarrow A'$ defined by $h_{\diamond}(se^{it}) = q_{\diamond}^{-1}(s)e^{it}$ is called the *critical Nitsche map*.

Notice that, the mapping

$$f^{\gamma}(se^{it}) = q(s)e^{it} : A' \rightarrow A$$

is the inverse of the harmonic diffeomorphism w .

Conjecture 9.2. [22] *Let ρ be a regular metric. If $r < 1$, and there exists a ρ -harmonic mapping of the annulus $A' = A(r, 1)$ onto the annulus $A = A(\delta, \sigma)$, then*

$$(9.7) \quad r \geq \exp \left(\int_{\sigma}^{\delta} \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - \delta^2 \rho^2(\delta)}} \right).$$

Notice that if $\rho = 1$, then this conjecture coincides with standard Nitsche conjecture and the latter conjecture is settled recently by Iwaniec, Kovalev and Onninen in [12]. For a regular metric we refer [21] for some partial solution of Conjecture 9.2.

In [22] it is proved that every Nitsche map $w(z) = p(r)e^{it}$ is a minimizing deformation and consequently a stationary deformation. In the following lemma we demonstrate the validity of Lemma 5.1 to this class of mappings.

Lemma 9.3. *For every Nitsche map $w = h^\gamma(z) = p(r)e^{it}$, where $z = re^{it}$ and $q(r) = p^{-1}(r)$ we have*

$$(9.8) \quad \text{Hopf}(w) = \frac{\gamma}{4z^2}.$$

In the notation of previous sections we have

$$(9.9) \quad \gamma = 4c.$$

Proof. Straightforward calculations yield that

$$(9.10) \quad w_z \overline{w_{\bar{z}}} = \frac{r^2 (p'(r))^2 - p(r)^2}{4z^2}.$$

Further by differentiating

$$r = \exp \left(\int_{\sigma}^{p(r)} \frac{dy}{\sqrt{y^2 + \gamma \varrho^2(y)}} \right)$$

we obtain

$$\frac{1}{r^2} = (p'(r))^2 \frac{\rho^2(p(r))}{\gamma + p^2(r)\rho^2(p(r))}.$$

Since

$$(9.11) \quad \text{Hopf}(w) = \rho^2(w(z))w_z \overline{w_{\bar{z}}},$$

we obtain (9.8). \square

Theorem 9.4. *Let $\Omega = A(r, 1)$ and $\Omega^* = A(R, 1)$ where $r < 1$ and $R < 1$. Let $r = e^{-\tau}$ and $\omega = e^{-R}$. Then $\tau = \text{Mod } \Omega$ and $\omega = \text{Mod } \Omega^*$. Let*

$$(9.12) \quad \Psi_\rho(\omega) := \int_R^1 \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - R^2 \rho^2(R)}}.$$

The function $\tau \mapsto \mathbf{E}^\rho(\tau, \omega)$ is C^2 -smooth on $(0, \infty)$, strictly convex for

$$\tau \leq \Psi_\rho(\omega)$$

and affine for

$$\tau > \Psi_\rho(\omega).$$

Notice that in this case we have

$$\tau_\diamond = \Psi_\rho(\text{Mod } \Omega^*) = \int_R^1 \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - R^2 \rho^2(R)}} > \log \frac{1}{R} = \text{Mod}(\Omega^*).$$

Before we proceed to the proof we recall that [15, Example 10.3] contains a special case for $\rho = 1$. The proof presented here is different and we believe it can be of more interest because it suggest the Conjecture 9.6.

Proof. Assume that there is a ρ -harmonic mapping between Ω and $\Omega^* = A(R, 1)$. We begin with the case

$$r \geq \exp \left(\int_1^R \frac{\rho(y) dy}{\sqrt{y^2 \rho^2(y) - R^2 \rho^2(R)}} \right).$$

By [22, Corollary 3.4] the absolute minimum of the energy integral

$$h \rightarrow E_\rho[h], \quad h \in W^{1,2}(A, A')$$

is attained by a ρ -Nitsche map

$$h^\gamma(z) = q^{-1}(s) e^{i(t+\beta)}, \quad z = s e^{it}, \quad \beta \in [0, 2\pi),$$

where

$$q(s) = \exp \left(\int_1^s \frac{dy}{\sqrt{y^2 + \gamma \varrho^2}} \right), \quad R < s < 1,$$

where $\gamma = \gamma(r)$ is defined by

$$(9.13) \quad r = \exp \left(\int_1^R \frac{dy}{\sqrt{y^2 + \gamma \varrho^2}} \right).$$

Further for $p(x) = q^{-1}(x)$ we compute

$$(9.14) \quad \mathcal{E}^\rho[h^\gamma] = 2\pi \int_{e^{-\tau}}^1 \frac{\gamma + 2\rho^2(p(t))p^2(t)}{t} dt,$$

which yields

$$(9.15) \quad \mathcal{E}^\rho[h^\gamma] = 2\pi \int_R^1 (\gamma(r) + 2\rho^2(s)s^2) \frac{q'(s)}{q(s)} ds.$$

Since

$$\frac{q'(s)}{q(s)} = \frac{1}{\sqrt{s^2 + \gamma(r)\varrho^2(s)}}$$

we obtain that

$$(9.16) \quad \mathcal{E}^\rho[h^\gamma] = 2\pi \int_R^1 \frac{\gamma(r) + 2\rho^2(s)s^2}{\sqrt{s^2 + \gamma(r)\varrho^2(s)}} ds$$

which implies

$$(9.17) \quad \epsilon(r) := \mathbb{E}^\rho(\log 1/r, \Omega^*) = 2\pi \int_R^1 \frac{\gamma(r) + 2\rho^2(s)s^2}{\sqrt{s^2 + \gamma(r)\varrho^2(s)}} ds.$$

It follows from (9.13) that γ is a differentiable function. By differentiating (9.17) with respect to r we get

$$(9.18) \quad \frac{d\epsilon(r)}{dr} = \pi \gamma(r) \gamma'(r) \int_R^1 \frac{\varrho^2(s) ds}{(s^2 + \gamma(r)\varrho^2(s))^{3/2}}.$$

But

$$(9.19) \quad \gamma'(r) \int_R^1 \frac{\varrho^2(s)ds}{(s^2 + \gamma(r)\varrho^2(s))^{3/2}} = \frac{2}{r}$$

and therefore

$$(9.20) \quad \frac{d\epsilon(r)}{dr} = \frac{2\pi\gamma(r)}{r}.$$

Thus

$$\frac{dE^\rho(\tau, \omega)}{d\tau} = -r \frac{2\pi\gamma(r)}{r}$$

i.e.

$$\frac{dE^\rho(\tau, \omega)}{d\tau} = -2\pi\gamma(r).$$

Further

$$(9.21) \quad \frac{d^2E^\rho(\tau, \omega)}{d\tau^2} = -r(-2\pi\gamma'(r)) = 2r\pi\gamma'(r).$$

Since $\gamma'(r) > 0$ for all r it follows that $E^\rho(\tau, \omega)$ is strictly convex under ρ -Nitsche condition. Observe that for $R = r$, $c(r) = 0$, and therefore

$$(9.22) \quad \frac{dE^\rho(\tau, \omega)}{d\tau} = 0.$$

Further for

$$r_\diamond = \exp(\tau_\diamond) = \exp\left(\int_1^R \frac{\rho(y)dy}{\sqrt{y^2\rho^2(y) - R^2\rho^2(R)}}\right),$$

i.e.

$$(9.23) \quad \gamma_\diamond := \gamma(r_\diamond) = -R^2\rho^2(R)$$

we have

$$(9.24) \quad \frac{dE^\rho(\tau, \omega)}{d\tau} = 2\pi R^2\rho^2(R)$$

and

$$\begin{aligned} \lim_{\tau \rightarrow \tau_\diamond - 0} \frac{d^2E^\rho(\tau, \omega)}{d\tau^2} &= \lim_{r \rightarrow r_\diamond} \pi\gamma'(r) \\ &= \lim_{r \rightarrow r_\diamond} \pi \left(\int_R^1 \frac{\varrho^2(s)ds}{(s^2 + \gamma(r)\varrho^2(s))^{3/2}} \right)^{-1} / (2r) \\ &= \left(\int_R^1 \frac{\varrho^2(s)ds}{(s^2 - R^2\rho^2(R)\varrho^2(s))^{3/2}} \right)^{-1} / (2r_\diamond) = 0, \end{aligned}$$

i.e.

$$(9.25) \quad \frac{d^2E^\rho(\tau, \omega)}{d\tau^2} \Big|_{\tau=\tau_\diamond-0} = 0.$$

It remains to consider the case

$$r < \exp \left(\int_1^R \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - R^2 \rho^2(R)}} \right).$$

Let

$$r_\diamond = \exp \left(\int_1^R \frac{\rho(y)dy}{\sqrt{y^2 \rho^2(y) - R^2 \rho^2(R)}} \right).$$

Then $r < r_\diamond < 1$. By [22, § 4.2.] the infimum $E^\rho(\Omega, \Omega^*)$ is realized by a non-injective deformation $h: \Omega \xrightarrow{\text{ontq}} \Omega^*$

$$h(z) = \begin{cases} R \frac{z}{|z|} & \text{for } r < |z| \leq r_\diamond \\ h_\diamond(z) & \text{for } r_\diamond \leq |z| < 1 \end{cases}$$

where $h_\diamond(z) = p_\diamond(|z|)e^{it} = (q_\diamond)^{-1}(|z|)e^{it}$, $z = |z|e^{it}$ and q_\diamond is defined in (9.6). Here the radial projection $z \mapsto Rz/|z|$ hammers $A(r, r_\diamond)$ onto the circle $|z| = R$ while the critical ρ -Nitsche mapping h_\diamond takes $A(r_\diamond, 1)$ homeomorphically onto Ω^* . The contribution of the radial projection to the energy of h is equal to

$$2\pi \int_r^{r_\diamond} \rho^2(R) R^2 \frac{dx}{x}$$

which together with contribution of h_\diamond gives

$$(9.26) \quad E^\rho(\Omega, \Omega^*) = -2\pi\gamma \int_r^{r_\diamond} \frac{dx}{x} + 2\pi \int_{r_\diamond}^1 \frac{\gamma + 2\rho^2(p_\diamond(t))p_\diamond^2(t)}{t} dt,$$

where $\gamma = -\rho^2(R)R^2$. This is an affine function of $\text{Mod}(\Omega) = \log 1/r$ whose first derivative equals $2\pi\rho^2(R)R^2$ and the second derivative vanishes. From (9.26) we have

$$\frac{dE^\rho(\tau, \omega)}{d\tau} \Big|_{\tau=\log \frac{1}{r_\diamond}} = 2\pi\rho^2(R)R^2 = -2\gamma = -8c.$$

Thus $E^\rho(\tau, \omega)$ is a C^2 -smooth function in $(0, \infty)$. □

Lemma 9.5. *The harmonic mapping $f^\gamma(z) = p^\gamma(s)e^{it}$ defined in (9.1) is quasiconformal minimizer if and only if $\gamma > -R^2\rho^2(R)$ or what is the same if and only if $\text{Mod}(A(r, 1)) < \tau_\diamond$, where τ_\diamond is defined in Theorem 9.4. If $\gamma = -R^2\rho^2(R)$, then the minimizer is harmonic but is not quasiconformal. The constant of quasiconformality is*

$$K_{f^\gamma} = \max \left\{ \sqrt{\frac{\gamma_\diamond - \gamma}{\gamma_\diamond}}, \sqrt{\frac{\gamma_\diamond}{\gamma_\diamond - \gamma}} \right\}.$$

Here $\gamma_\diamond = -R^2\rho^2(R)$ (see (9.23)).

Proof. First of all

$$\mu_f(z) = \frac{|sp'(s) - p(s)|}{sp'(s) + p(s)}.$$

From

$$\frac{1}{s^2} = (p'(r))^2 \frac{\rho^2(p(r))}{\gamma + p^2(s)\rho^2(p(s))},$$

we obtain

$$\mu_f(z) = \left| \frac{-p(s)\rho(p(s)) + \sqrt{\gamma + p^2(s)\rho^2(p(s))}}{p(s)\rho(p(s)) + \sqrt{\gamma + p^2(s)\rho^2(p(s))}} \right|,$$

and therefore $\mu_f(z) < 1$ if and only if $\gamma + p^2(s)\rho^2(p(s)) > 0$ for $R \leq s \leq 1$ and therefore for $\gamma > -R^2\rho^2(R)$. Moreover

$$K_f(z) = \frac{1 + \mu_f(z)}{1 - \mu_f(z)} = \max \left\{ \frac{\sqrt{\gamma + p^2(s)\rho^2(p(s))}}{p(s)\rho(p(s))}, \frac{p(s)\rho(p(s))}{\sqrt{\gamma + p^2(s)\rho^2(p(s))}} \right\}$$

and thus

$$K_f := \max_z K_f(z) = \max \left\{ \frac{\sqrt{\gamma + p^2(R)\rho^2(p(R))}}{p(R)\rho(p(R))}, \frac{p(R)\rho(p(R))}{\sqrt{\gamma + p^2(R)\rho^2(p(R))}} \right\},$$

which can be written as

$$K_f = \max \left\{ \sqrt{\frac{\gamma_\diamond - \gamma}{\gamma_\diamond}}, \sqrt{\frac{\gamma_\diamond}{\gamma_\diamond - \gamma}} \right\},$$

where $\gamma = 4c(\tau)$ and $\gamma_\diamond = 4c(\tau_\diamond)$. Thus the minimum of K_f is for $\gamma = 0$, i.e. for $\tau = \text{Mod}(\Omega) = \text{Mod}(\Omega^*)$ and is increasing for $\gamma > 0$ i.e. $\tau < \text{Mod}(\Omega^*)$ and decreasing for $\gamma < 0$ i.e. $\tau > \text{Mod}(\Omega^*)$. This shows that K_f depends only on $\text{Mod}(\Omega)$ and $\text{Mod}(\Omega^\circ)$. \square

Lemma 9.5 is a motivation for the following conjecture

Conjecture 9.6. a) If f is a minimizer of the energy \mathcal{E}^ρ between two doubly connected domains Ω , ($\tau = \text{Mod}(\Omega)$) and Ω^* , then f is harmonic and $K(\tau)$ -quasiconformal if and only if τ is smaller than the modulus τ_\diamond of critical Nitsche domain $A(\tau_\diamond)$ (see Remark 8.7 and Corollary 8.6).

b) Under condition of a) we conjecture that $\gamma_\diamond = 4c_\diamond < 0$ (see Remark 8.7) and

$$(9.27) \quad K(\tau) = \max \left\{ \sqrt{\frac{\gamma_\diamond - \gamma}{\gamma_\diamond}}, \sqrt{\frac{\gamma_\diamond}{\gamma_\diamond - \gamma}} \right\},$$

where $\gamma = 4c(\tau)$ and $\gamma_\diamond = 4c(\tau_\diamond)$.

Concerning Conjecture 9.6 we offer the following observation

Remark 9.7. a) As every minimizer is unique up to conformal transformation of the domain, provided that it is a homeomorphism, it follows from Lemma 8.6 that Conjecture 9.6 holds true if the image domain is a circular annulus and ρ is a radial metric. In this case the solution of minimization problem for L^1 -norm of distortion between circular annuli is the quasiconformal mapping $f_1(z) = q^\gamma(|z|)e^{i \arg z}$. If instead of L^1 -norm of distortion we consider L^∞ -norm of distortion, then the minimum is attained for the mapping $f_\infty(z) = |z|^{\log r / \log R} e^{i \arg z}$. As f_1 and f_∞ are quasiconformal mappings

between annuli $A(R, 1)$ and $A(r, 1)$, it follows from quasiconformal theory that

$$K_{f_\infty} = \max \left\{ \frac{\log R}{\log r}, \frac{\log r}{\log R} \right\} \leq K_{f_1}$$

with equality if and only if $\rho(z) = 1/|z|$ or $\gamma = 0$.

b) It follows from (5.9) that for every stationary transformation h we have

$$\mathbb{K}_h := \frac{1}{2} \left(K_h + \frac{1}{K_h} \right) = \frac{\|Dh\|^2}{2J_h} = \frac{|h_N|^2 + |h_T|^2}{2|h_N| \cdot |h_T|}.$$

Combining with (5.7) we obtain

$$\mathbb{K}_h = \frac{2|h_N|^2 - \frac{4c}{\rho^2(h(z))|z|^2}}{2\sqrt{|h_N|^2(|h_N|^2 - \frac{4c}{\rho^2(h(z))|z|^2})}}.$$

Therefore for $\Gamma = \rho^2(h)|h_N|^2|z|^2$ and $\gamma = 4c$

$$K_h(z) = \max \left\{ \sqrt{\frac{\Gamma - \gamma}{\Gamma}}, \sqrt{\frac{\Gamma}{\Gamma - \gamma}} \right\}.$$

As $c(\tau_\diamond) \leq 0$, it can be shown that $K_h(z) \leq K(\tau)$ if and only if

$$\rho^2(h)|h_N|^2|z|^2 \geq 4(c(\tau) - c(\tau_\diamond))$$

or what is the same if and only if

$$\rho^2(h)|h_T|^2|z|^2 \geq -4c(\tau_\diamond).$$

We believe that the inequality (5.6) above can be used in this context.

c) It has been shown in some recent papers that every q.c. harmonic mapping between domains with smooth boundaries, say $C^{1,\alpha}$ is bi-Lipschitz continuous (see e.g. [23]-[25], [32], [30]). If Conjecture 9.6 is true, then every minimizer between smooth double connected domains is a bi-Lipschitz mapping provided that $\tau < \tau_\diamond$.

9.1. Appendix. Here we give two important metrics for which the results of this section can be stated in a more explicit way.

Example 9.8. Let ρ be the Riemann metric $\rho = \frac{2}{1+|z|^2}$. Equation (1.1) becomes

$$(9.28) \quad u_{z\bar{z}} - \frac{2\bar{u}}{1+|u|^2} u_z \cdot u_{\bar{z}} = 0.$$

Notice this important example. The Gauss map of a surface Σ in \mathbb{R}^3 sends a point on the surface to the corresponding unit normal vector $\mathbf{n} \in \overline{\mathbb{C}} \cong S^2$. In terms of a conformal coordinate z on the surface, if the surface has *constant mean curvature*, its Gauss map $\mathbf{n} : \Sigma \mapsto \overline{\mathbb{C}}$, is a Riemann harmonic map [33]. Since

$$\int_{\mathbb{C}} \rho^2(w) du dv = 4\pi,$$

it follows that the Riemann metric is allowable for every double connected domain (bounded or unbounded).

Example 9.9. If $u : \mathbb{U} \mapsto \mathbb{U}$ is a harmonic mapping with respect to the hyperbolic metric $\lambda = \frac{2}{1 - |z|^2}$ then Euler-Lagrange equation of u is

$$(9.29) \quad u_{z\bar{z}} + \frac{2\bar{u}}{1 - |u|^2} u_z \cdot u_{\bar{z}} = 0.$$

An important example of hyperbolic harmonic mapping is the Gauss map of a space-like surfaces with constant mean curvature H in the Minkowski 3-space $M^{2,1}$ (see [4]). This metric is allowable in compact bounded domains in \mathbb{U} but for every $r < 1$, the integral $\int_{A(r,1)} \lambda^2(w) du dv$ diverges.

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FACULTY OF NATURAL SCIENCES AND MATHEMATICS, UNIVERSITY OF MONTENEGRO,
 CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO
E-mail address: `davidk@ac.me`